HYPERBOLIC POLYNOMIALS AND MULTIPARAMETER REAL ANALYTIC PERTURBATION THEORY

KRZYSZTOF KURDYKA AND LAURENTIU PAUNESCU

ABSTRACT. Let $P(x,z) = z^d + \sum_{i=1}^d a_i(x)z^{d-i}$ be a polynomial, where a_i are real analytic functions in an open subset U of $\mathbb{R}^n$. If for any $x \in U$ the polynomial $z \mapsto P(x,z)$ has only real roots, then we can write those roots as locally lipschitz functions of x. Moreover, there exists a modification (a locally finite composition of blowing-ups with smooth centers) $\sigma : W \to U$ such that the roots of the corresponding polynomial $\tilde{P}(w,z) = P(\sigma(w),z)$, $w \in W$, can be written locally as analytic functions of w. Let $A(x)$, $x \in U$ be an analytic family of symmetric matrices, where U is open in $\mathbb{R}^n$. Then there exists a modification $\sigma : W \to U$, such the corresponding family $\tilde{A}(w) = A(\sigma(w))$ can be locally diagonalized analytically (i.e. we can choose locally a basis of eigenvectors in an analytic way). This generalizes the Rellich's well known theorem (1937) for one parameter families. Similarly for an analytic family $A(x)$, $x \in U$ of antisymmetric matrices there exits a modification σ such that we can find locally a basis of proper subspaces in an analytic way.

1. Introduction

In the late 30's F. Rellich [27],[28] developed the theory of one parameter analytic perturbation theory of linear operators. This theory culminates with the celebrated monograph of T. Kato [14]. To study the behaviour of eigenvalues of symmetric matrices under analytic one-parameter perturbation Rellich proved the following fundamental fact. Let

$$P(x,z) = z^d + \sum_{i=1}^{d} a_i(x)z^{d-i} \tag{1.1}$$

be a polynomial, where a_i are real analytic functions on an open interval $I \subset \mathbb{R}$. If for any $x \in I$ the polynomial $z \mapsto P(x,z)$ has only real roots (we call such a polynomial *hyperbolic*), then there are analytic functions $f_i : I \to \mathbb{R}$, $i = 1, \dots, n$ such that $P(x,z) = \prod_{i=1}^d (z - f_i(x))$. In other words we can choose analytically the roots of P.

If we consider a multiparameter version of this theorem, i.e. we assume now that a_i are real analytic functions in an open subset U of $\mathbb{R}^n$, $n > 1$, then we have a simple counterexample $z^2 - (x_1^2 + x_2^2)$. For this reason a multiparameter perturbation theory was not developed (to our knowledge), though it was suggested by Rellich. In this paper we give some generalizations

Date: 21 January 2006.

1991 *Mathematics Subject Classification.* 15A18,32B20,14P20.

Key words and phrases. real analytic, subanalytic, arc-analytic, lipschitz.

The first author thanks University of Sydney for support. Part of this work was done while the second author was visiting MSRI Berkeley.

of Rellich's theory in the multiparameter case. These generalizations are purely real, they make no sense in the complex case developed by Kato.

The first generalization was inspired by S. Łojasiewicz, who suggested that the roots of the polynomial P can be chosen locally in a lipschitz way. This is true as we prove in Theorem 4.1. The result is quite delicate since in the one parameter case (the Rellich Theorem) there is no way to bound the lipschitz constant for roots in terms of bounds for the coefficients $a_i(x)$. The proof is obtained by a reduction to the 2-parameter case and by a careful study of a desingularization of singularities of the zeros of P. In fact we are able to keep track of partial derivatives of roots after blowing up, because we are dealing with a family of hyperbolic polynomials. This is rather surprising, since it is known that this is impossible in general, for instance there are blow-analytic (or arc-analytic functions) which are not locally lipschitz. Our result is related to Lidski's Theorem which implies that the spectral mapping on the space of symmetric matrices is lipschitz (globally). More precisely, our Theorem 4.1 implies the above corollary of Lidskii's theorem (in a weaker form), however Lidskii's theorem does not imply our theorem. Indeed as Rellich noticed in [28] not every analytic family of hyperbolic polynomials can be written as characteristic polynomials of an analytic family of symmetric matrices. Surprisingly this is related to Hilbert's 17th problem for analytic functions. In a similar way we prove an analogue of Theorem 4.1 in the case where all the roots of $z \mapsto P(x,z)$ are purely imaginary (we call such a polynomial *antihyperbolic*). This is important in the study of analytic families of antisymmetric matrices.

The second direction of generalization is related to the theory of arc-analytic functions; initiated by the first author in [17]. Actually the roots of an analytic family of hyperbolic polynomials can be seen as a multivalued arc-analytic function. As we prove in Theorem 6.11 it turns out that after suitable blowing-ups of the space of parameters we can write locally the roots of hyperbolic polynomials as analytic functions of parameters.

Rellich's theory deals not only with eigenvalues but also with eigenvectors. He proved (see for instance [28]) that every one parameter analytic family of symmetric matrices admits an analytic choice of bases of eigenvectors. In other words such a family can be analytically diagonalized even if the eigenvalues become multiple. As we prove in Theorem 7.2 this can be also done for multiparameter analytic families, but first we have to blow up the space of parameters in order to make the eigenvalues locally normal crossing.

Finally, we also study analytic families of antisymmetric matrices depending on several parameters. We prove analogously that, after suitable blowing-ups of the parameters, we can reduce them locally to the canonical form in an analytic way.

2. Arc-analytic functions

For further convenience we recall here some facts concerning arc-analyticity. Let U be an open subset of $\mathbb{R}^n$. Following [17] we say that a map $f : U \to \mathbb{R}^k$ is *arc-analytic* if for any analytic arc $\alpha : (-\varepsilon, \varepsilon) \to U$, the composed function $f \circ \alpha$ is also analytic.

In general arc-analytic maps are very far from being analytic, in particular there are arc-analytic functions which are not subanalytic [18], not continuous [7], with a non-discrete singular set [19]. Hence it is natural to consider only arc-analytic maps with subanalytic graphs. Earlier T.-C. Kuo [16], motivated by equisingularity problems, introduced the notion

of *blow-analytic* functions, i.e. functions which become analytic after a composition with appropriate proper bimeromorphic maps (e.g. a composition of blowing-ups with smooth centers). Clearly any blow-analytic mapping is arc-analytic and subanalytic. The converse holds in a slightly weaker form [6], see also [24]. We shall explain it in the next section.

Blow-analytic maps have been studied by several authors (see the survey [9]). It is known that in general subanalytic and arc-analytic functions are continuous [17], but not necessarily (locally) lipschitz [9], [26].

The following examples are arc-analytic but not analytic functions:

$$f = \frac{x^3}{x^2+y^2},\ g = \frac{xy^5}{x^4+y^6},\ h = \sqrt{x^4+y^4}.$$

The function f is locally lipschitz, but not C^1 (cf. [17]), the function g is not locally lipschitz (cf. [26]). The function h is C^1 but not C^2 (cf. [6]).

Recently we have proved in [20] that, if h is arc-analytic and h^r is analytic for some integer r, then h is locally lipschitz. However arc-analytic roots of polynomials with analytic coefficients are not necessarily lipschitz.

Example 2.1. Consider a polynomial $P(x,y,z) = (z^4 - (x^2+y^8))^2 - x^4 - y^{20}$. It has an arc-analytic root

$$f = \sqrt[4]{x^2+y^8-\sqrt{x^4+y^{20}}},$$

which is not lipschitz! Note that the above polynomial is not hyperbolic.

It is useful to consider arc-analytic complex valued functions, where we understand that they are analytic on real analytic arcs. We cannot avoid arc-analytic solutions in the sense above. Indeed we have the following type of examples which appear in our context:

Example 2.2. Consider $P(z,x,y) = z^4 - x^8 - y^8$ as polynomial in z, then it has the obvious roots:

$$z_1 = \sqrt[4]{x^8+y^8}, z_2 = -\sqrt[4]{x^8+y^8}, z_3 = i\sqrt[4]{x^8+y^8}, z_4 = -i\sqrt[4]{x^8+y^8}.$$

2.1. **Locally blow-analytic functions.** We recall some of the notions used in this paper (for more information see for instance [7], [9], [10], [16], [18], [19], [25]).

We recall first a definition of a local blowing-up. Let M be an analytic manifold and $\Omega \subset M$ an open set. Assume that X is an analytic submanifold of M, closed in Ω. Then we can define $\tau : \tilde{\Omega} \to \Omega$, the blowing-up of Ω with the center X, see for instance [13] or [23]. A restriction of τ to an open subset of $\tilde{\Omega}$ is called *a local blowing-up with a smooth (nowhere dense) center.*

Let U be a neighbourhood of the origin of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ denote a map defined on U except possibly some nowhere dense subanalytic subset of U. We say that f is *locally blow-analytic* via a locally finite collection of analytic modifications $\sigma_\alpha : W_\alpha \to \mathbb{R}^n$, if for each α we have

(i) W_α is isomorphic to $\mathbb{R}^n$ and σ_α is the composition of finitely many local blowing-ups with smooth nowhere dense centers, and $f \circ \sigma_\alpha$ has an analytic extension on W_α.

(ii) There are subanalytic compact subsets $K_\alpha \subset W_\alpha$ such that $\bigcup \sigma_\alpha(K_\alpha)$ is a neighbourhood of $\overline{U}$.

The notion of *(locally) blow-analytic* functions (or maps) is very much related to the notion of *arc-analytic* functions. Indeed in [6], see also [24], it is proved that an *arc-analytic* function has *subanalytic* graph if and only if it is *locally blow-analytic*.

Remark 2.3. . The definition of arc-analytic function is much more intrinsic and it is usually easier to check that a given function is arc-analytic than to check that it is blow-analytic. Actually, when the first author introduced (in mid 80's) arc-analytic functions he has hoped that subanalytic and arc-analytic are exactly the same with (globally) blow-analytic. This is true for semialgebraic functions and for functions in 2 variables (since we blow up only points). In a forthcoming paper (joint with A. Parusiński) the authors shall give a proof of this conjecture for functions in 3 variables. But the general case presents serious difficulties and remains still open.

3. Hyperbolic Polynomials

3.1. **Splitting lemma for polynomials.** Given a p-tuple $a = (a_1, \dots, a_p) \in \mathbb{R}^p$ and a q-tuple $b = (b_1, \dots, b_q) \in \mathbb{R}^q$, we associate two polynomials

$$P_a(z) = z^p + \sum_{i=1}^{p} a_i z^{p-i}, \;\; Q_b(z) = z^q + \sum_{j=1}^{q} b_j z^{q-j}.$$

We consider the product of these polynomials

$$P_a Q_b = R_c = z^{p+q} + \sum_{k=1}^{p+q} c_k z^{p+q-k},$$

where $c = (c_1, \dots, c_{p+q}) \in \mathbb{R}^{p+q}$. This defines a polynomial map

$$\Phi : \mathbb{R}^p \times \mathbb{R}^q \ni (a, b) \mapsto c \in \mathbb{R}^{p+q}.$$

The following lemma is crucial.

Lemma 3.1. (Hensel's Splitting Lemma)

(i) *The jacobian of Φ at (a, b) is equal (up to sign) to the resultant of P_a and Q_b.*

(ii) *Let us fix $\bar{a} = (\bar{a}_1, \dots, \bar{a}_k) \in \mathbb{R}^p$ and $\bar{b} = (b_1, \dots, \bar{b}_q) \in \mathbb{R}^q$ and assume that corresponding polynomials $P_{\bar{a}}$ and $Q_{\bar{b}}$ have no common zeros in $\mathbb{C}$, in other words that their resultant is non zero. Then there exists a neighbourhood $U \subset \mathbb{R}^{p+q}$ of $\bar{c} = \Phi(\bar{a}, \bar{b})$ such that for any $c \in U$ the corresponding polynomial splits in a unique way, $R_c = P_a Q_b$, moreover the mapping $a = a(c), b = b(c)$ is analytic (even Nash) and satisfies $\bar{a} = a(\bar{c}), \bar{b} = b(\bar{c})$.*

Indeed, it is easy to see that the Jacobian matrix of Φ is exactly the Sylvester matrix of the pair P_a and Q_b. So the resultant $Res(P_a, Q_b)$, which is by the definition equal to (up to sign) the determinant of the Sylvester matrix, is also equal to the jacobian of Φ at (a, b), see e.g. [6],[1]. Recall that two polynomials P_a and Q_b have no common zeros in $\mathbb{C}$, if and only

if their resultant is non zero. The second part of the lemma is just a consequence of the fact that Φ is invertible in a neighbourhood of (a, b), in particular Φ^{-1} is analytic (even Nash) by the Inverse Mapping Theorem.

In the sequel we will use the following consequence of the splitting Lemma.

Corollary 3.2. *Let $R(x, z) = z^r + \sum_{k=1}^r c_k(x) z^{r-k}$, where $c_k(x)$ are analytic functions in some open set $\Omega \subset \mathbb{R}^m$. Assume that for some $x_0 \in \Omega$ the polynomial $z \mapsto R(x_0, z)$ splits, i.e. $R(x_0, z) = P_{x_0}(z) Q_{x_0}(z)$, where $\deg P_{x_0} = p$, $\deg Q_{x_0} = q$ and $r = p + q$. Suppose moreover that $P_{x_0}(z)$ and $Q_{x_0}(z)$ have no common roots in $\mathbb{C}$. Then there exist a neighbourhood $U \subset \Omega$ of x_0, and analytic functions, $a_i : U \to \mathbb{R}$, $i = 1, \ldots, p$ and $b_j : U \to \mathbb{R}$, $j = 1, \ldots, q$ such that*

$$R(x, z) = P(x, z) Q(x, z),\ x \in U,\ z \in \mathbb{R},$$

where $P(x, z) = z^p + \sum_{i=1}^p a_i(x) z^{p-i}$, $Q(x, z) = z^q + \sum_{j=1}^q b_j(x) z^{q-j}$. Moreover $P(x_0, z) = P_{x_0}(z)$ and $Q(x_0, z) = Q_{x_0}(z)$.

Remark. Splitting Lemma and Corollary 3.2 hold of course over complex numbers, but we don't need this.

3.2. **Newton-Puiseux Expansions.** For latter use we recall some classical facts about the roots of Weierstrass polynomials. Let

$$P(x, z) = z^d + \sum_{i=1}^d a_i(x) z^{d-i}, \tag{3.1}$$

with a_i real analytic functions in a neighbourhood of $0 \in \mathbb{R}$. Then there are holomorphic functions h_i, $i = 1, \ldots, d$ and an integer r such that

$$P(x, z) = \prod_{i=1}^d [z - h_i(x^{1/r})] = \prod_{i=1}^d [z - f_i(x)]$$

for $x \geq 0$ close enough to 0, and any $z \in \mathbb{C}$. We call $f_i(x) = h_i(x^{1/r})$ a *Newton-Puiseux root of P*. Clearly each f_i is given by a *Puiseux expansion* $f_i(x) = \sum_{\nu=0}^{\infty} \alpha_\nu^i x^{\nu/r}$.

3.3. **Hyperbolic polynomials.** Let

$$P(z) = z^d + \sum_{i=1}^d a_i z^{d-i}$$

be a polynomial with real coefficients. Let $z_1, \ldots, z_d$ be all complex roots of P; recall that $a_1 = z_1 + \cdots + z_d$. By *Tschirnhausen transformation*, which is the change of variable $z \mapsto z - \frac{a_1}{d}$, we may assume that $a_1 = 0$. We say that P *is hyperbolic* if all its roots are real. Hyperbolic polynomials appear naturally, for instance as characteristic polynomials of symmetric matrices. We state now two elementary but crucial properties of hyperbolic polynomials.

Lemma 3.3. *Let $P(z) = z^d + \sum_{i=2}^d a_i z^{d-i}$ be a polynomial with real coefficients (note that $a_1 = 0$). Denote the roots (possibly complex) of P by $z_1, \ldots, z_d$. Then*

$$z_1^2 + \cdots + z_d^2 = -2a_2. \tag{3.2}$$

Consequently, if P is hyperbolic, then $a_1 = a_2 = 0$ if and only if $P(z) = z^d$, that is 0 is the only root of P.

Proof: since $a_2 = \sum_{i<j} z_i z_j$, we have $z_1^2 + \cdots + z_d^2 = a_1^2 - 2a_2 = -2a_2$. If all z_i are real, then $z_1^2 + \cdots + z_d^2 = 0$ implies that all $z_i = 0$.

In the sequel we will study families of monic polynomials depending analytically on parameters (i.e. the coefficients are analytic functions of parameters), such that for each values of the parameters the corresponding polynomial is hyperbolic. We will also call, for short, such a family *a hyperbolic polynomial*.

3.4. **Rellich's Theorem.** In the late 30's Rellich [27] proved a rather surprising fact about the roots of hyperbolic polynomials of the form (3.1). His result is the following.

Theorem 3.4. (Rellich 1937) *Consider a polynomial*

$$P(x,z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i},$$

with a_i real analytic functions in an open interval $I \subset \mathbb{R}$. Assume that for each $x \in I$ all the roots of the polynomial $z \mapsto P(x,z)$ are real. Then there exist real analytic functions $f_i : I \to \mathbb{R}$ such that

$$P(x,z) = \prod_{i=1}^{d} [z - f_i(x)],\ x \in I, z \in \mathbb{R}. \tag{3.3}$$

We will outline a proof of the above theorem.

It is inspired by [1], however we made it shorter since we use Puiseux's theorem.

Note that by the analytic extension argument it is enough to prove the theorem locally. We fix a point, say $0 \in I$, and assume that a_i are analytic in a neighbourhood of 0.

- 1st Step:

  We may assume that all $a_i(0) = 0$.

  Indeed, if $P(0,z) = (z-c)^d$ then, shifting $z \mapsto z - c$, we may assume that $c = 0$. Consequently all $a_i(0) = 0$. Otherwise $P(0,z) = (z-c)^p P_2(z)$, $0 < p < d$, with $P_2(c) \neq 0$. Applying Corollary 3.2 we can split our polynomial as $P(x,z) = P_1(x,z)P_2(x,z)$, where P_1 and P_2 are of the form (3.1) with real analytic coefficients in a neighbourhood of $0 \in \mathbb{R}$. Hence we can handle separately P_1 which is already of the form considered above, and P_2 which is of a smaller degree.
- 2nd Step:

  Let us write

  $$P(x,z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i} = \prod_{i=1}^{d} [z - f_i(x)],\ x > 0,$$

  where all a_i are real analytic in a neighbourhood of $0 \in \mathbb{R}$, and $a_i(0) = 0$, hence also $f_i(0) = 0$. Applying Tschirnhausen transformation $z \mapsto z - \frac{a_1(x)}{d}$ we may assume

that $a_1(x) \equiv 0$. Denote by $z_1(x), \dots, z_d(x)$ all the roots of $z \mapsto P(x,z)$. Then Lemma 3.3 yields

$$-2a_2(x) = z_1(x)^2 + \cdots + z_d(x)^2, \tag{3.4}$$

for x in a neighbourhood of $0 \in \mathbb{R}$. But by our assumption all the roots $z_i(x)$ are real, hence a_2 must be negative. Consequently the order of a_2 at 0 is even and we can write

$$a_2(x) = x^2 b(x), \tag{3.5}$$

with $b(x)$ analytic in a neighbourhood of $0 \in \mathbb{R}$. Applying (3.4) and (3.5) to the Puiseux roots of P we obtain

$$f_1(x)^2 + \cdots + f_d(x)^2 = x^2 b(x). \tag{3.6}$$

So we easily deduce the following lemma.

Lemma 3.5. *The order of each f_i at 0 is greater or equal than 1.*

By the order of f_i we mean the smallest rational exponent in its Puiseux expansion such that its coefficient does not vanish.

Accordingly, by Viéte's formulas $a_i = (-1)^i \sum\limits_{k_1 < \cdots < k_i} z_{k_1} \dots z_{k_i}$, we obtain:

Lemma 3.6. *The order of each a_i at 0 is greater or equal than i.*

Now we are in the position to conclude Rellich's theorem. We are going to show that in the Puiseux expansion of each f_i there are only integer exponents.
Let us write $f_i(x) = \sum_{\nu=0}^{\infty} \alpha_\nu^i x^{\nu/r}$. By Lemma 3.5 we know that all $\alpha_1^i = \cdots = \alpha_{r-1}^i = 0$, so

$$\frac{f_i(x)}{x} = \sum_{\nu=r}^{\infty} \alpha_\nu^i x^{\nu/r-1}$$

are all bounded, and they are the Puiseux roots of the polynomial

$$\tilde{P}(x,z) = z^d + \sum_{i=1}^{d} \frac{a_i(x)}{x^i} z^{d-i},$$

with $\tilde{a}_i(x) = \frac{a_i(x)}{x^i}$ real analytic at $0 \in \mathbb{R}$, by Lemma 3.6. Now we apply the first step of the reduction to the polynomial $\tilde{P}(x,z)$. So may assume that all $\tilde{a}_i(0) = 0$. Note that the shift affects only the coefficient α_r^i. By Lemma 3.5 we deduce that

$$\alpha_\nu^i = 0, \; \nu = r+1, \dots, 2r-1.$$

Continuing this process we see that for any integer ν which is not a multiple of r, the corresponding coefficient vanishes, that is $\alpha_\nu^i = 0$. Actually we can write f_i as a convergent power series $f_i(x) = \sum_{n=0}^{\infty} \alpha_{rn}^i x^n$.

Formally our argument applies only for $x > 0$, but since now we know that f_i are analytic in a neighbourhood of $0 \in \mathbb{R}$, we deduce that also for negative x, $f_1(x), \dots, f_d(x)$ are the

roots of the polynomial $z \mapsto P(x,z)$. So we can write

$$P(x,z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i} = \prod_{i=1}^{d} [z - f_i(x)]$$

with f_i analytic in a neighbourhood of $0 \in \mathbb{R}$. By the analytic extension argument, each f_i extends to a unique analytic function on the whole interval I. Hence Rellich's theorem follows.

3.5. **Expansions of the roots of hyperbolic polynomials in 2 parameters.** Consider a polynomial

$$P(x,y,z) = z^d + \sum_{i=1}^{d} a_i(x,y) z^{d-i}, \tag{3.7}$$

with $a_i(x,y)$ analytic in a neighbourhood of $0 \in \mathbb{R}^2$.

Proposition 3.7. *Assume that P is hyperbolic with respect to z, that is for each (x,y), the polynomial $z \mapsto P(x,y,z)$ has only real roots. Then, in a set $H = \{|x| < y^N,\ 0 < y < \delta\}$, where N is large enough and δ is small enough, the polynomial P splits in the form*

$$P(x,y,z) = \prod_{i=1}^{d} [z - f_i(x,y)],\ (x,y) \in H,\ z \in \mathbb{R}, \tag{3.8}$$

with f_i are analytic in H.

Now we can state a key proposition, which allows us to prove the fact that the roots of hyperbolic polynomials are lipschitz.

Proposition 3.8. *Each function $y \mapsto \frac{\partial f_i}{\partial x}(0,y)$ extends to an analytic function in a neighbourhood of $0 \in \mathbb{R}$, in particular $\frac{\partial f_i}{\partial x}(0,y)$ are bounded for $y \in (0,\delta)$.*

Proof of Proposition 3.7. We shall proceed by the induction on the highest multiplicity of a root of the univariate polynomial $z \mapsto P(0,0,z)$.

- Case 0. All roots of $z \mapsto P(0,0,z)$ are simple, then the statement of the proposition is an immediate consequence of the Implicit Function Theorem.
- Case 1. Let c be a root of $z \mapsto P(0,0,z)$ of the maximal multiplicity, We can suppose that $P(0,0,z) = (z-c)^d$. Otherwise $P(0,0,z) = (z-c)^p P_2(z)$, $0 < p < d$, with $P_2(c) \neq 0$. Applying Corollary 3.2 we can split our polynomial as $P(x,y,z) = P_1(x,y,z) P_2(x,y,z)$, where P_1 and P_2 are of the form (3.7) with real analytic coefficients in a neighbourhood of $0 \in \mathbb{R}$. Hence we can handle separately P_1, which is already of the form considered above, and P_2 which is of a smaller degree. Finally, shifting $z \mapsto z - c$, we may suppose that $c = 0$.

  So in formula (3.7) we may suppose that all $a_i(0,0) = 0$.

  Now we consider the hyperbolic polynomial $P(0,y,z)$. According to Rellich's Theorem 3.4 we have

$$P(0,y,z) = \prod_{i=1}^{d} [z - c_0^i(y)]$$

with $c_0^i(y)$ analytic in a neighbourhood of $0 \in \mathbb{R}$.

- Case 1.1. Assume that not all $c_0^i(y)$ are identical as functions, note that $c_0^i(0) = 0$ for all i. We are going to describe an operation, which will allow us to reduce the multiplicity of the root $c = 0$. Let f, g be two distinct analytic functions in a neighbourhood of $0 \in \mathbb{R}$, then

$$f(y) - g(y) = y^k b(y)$$

where $b(y)$ is analytic and $b(0) \neq 0$. We will call k the *order of contact of* f and g at 0.

We consider a privileged chart of the blowing-up of the origin in $\mathbb{R}^2$, more precisely the mapping $\sigma : (x, y) \mapsto (xy, y)$. Note that by Lemma 3.6 we have

$$a_i(xy, y) = \tilde{a}_i(x, y) y^i, \tag{3.9}$$

with $\tilde{a}_i(x, y)$ analytic in a neighbourhood of $0 \in \mathbb{R}^2$. We put

$$\tilde{P}(x, y, z) = z^d + \sum_{i=1}^{d} \tilde{a}_i(x, y) z^{d-i}. \tag{3.10}$$

We call $\tilde{c}_0^i(y) = \frac{c_0^i(y)}{y}$ the *proper transform of* $c_0^i(y)$, they are the roots of the polynomial

$$\tilde{P}(0, y, z) = z^d + \sum_{i=1}^{d} \tilde{a}_i(0, y) z^{d-i}. \tag{3.11}$$

Clearly the orders of contact between the above proper transforms drop by 1. Either all $\tilde{c}_0^i(y)$ take the same value at the origin, so by a shift we may assume that they vanish at 0 and we continue our procedure, or the highest multiplicity of the roots of the polynomial $\tilde{P}(0, 0, z)$ decreases. Note that the above procedure has to be finite, as there are at least two distinct roots $c_0^i \neq c_0^j$.

To conclude the first part of the Proposition 3.7 we have to explain the remaining case.

- Case 1.2. Assume that $c_0^1(y) = \cdots = c_0^d(y)$. By shifting we may assume that $c_0^1(y) = \cdots = c_0^d(y) \equiv 0$.

Note that $a_i(0, y) \equiv 0$ for all i. After Tschirnhausen transformation we may assume that $a_1(x, y) \equiv 0$.

Now we consider the coefficient a_2; either $a_2(x, y) \equiv 0$ and then, by Lemma 3.3, polynomial P has the only root $z(x, y) \equiv 0$ and we are done, or $a_2(x, y) \not\equiv 0$. In the second case there exists an integer k such that

$$y \mapsto \frac{\partial^k a_2}{\partial x^k}(0, y) \not\equiv 0, \tag{3.12}$$

does not vanish identically. We take the smallest such an integer. By the equation (3.5) k must be even, so we write $k = 2r$.

Applying Lemma 3.6 to our polynomial with y fixed, we obtain that

$$a_i(x, y) = \tilde{a}_i(x, y) x^{ri}, \tag{3.13}$$

with $\tilde{a}_i(x,y)$ analytic in a neighbourhood of $0 \in \mathbb{R}^2$. Now we consider the polynomial

$$\tilde{P}(x,y,z) = z^p + \sum_{i=2}^{p} \tilde{a}_i(x,y) z^{p-i}, \tag{3.14}$$

Note that, by (3.12), we know that $y \mapsto \tilde{a}_2(0,y) \not\equiv 0$. As a consequence, by Lemma 3.3, the polynomial $(y,z) \mapsto \tilde{P}(0,y,z)$ has at least 2 distinct roots $\tilde{c}_0^i(y)$ so we may apply the argument of the Case 1.1 and we are done by the induction on the highest multiplicity.

So we have proved the following: there exists an integer N such that

$$P(xy^N, y, z) = z^d + \sum_{i=1}^{d} a_i(xy^N, y) z^{d-i} = \prod_{i=1}^{d} [z - g_i(x,y)],$$

where g_i are analytic in a neighbourhood of $0 \in \mathbb{R}^2$.

So $f_i(x,y) = g_i(xy^{-N}, y)$ are the functions we claimed in Proposition 3.7.

We have a more precise control of the functions f_i. Replacing f_i by $f_i(x,y) - f_i(0,y)$ we may assume that $f_i(0,y) \equiv 0$. (Note that $(x,y) \mapsto f_i(0,y)$ is analytic in a neighbourhood of $0 \in \mathbb{R}^2$.) Now we can define a strict transform of f_i as

$$f_i^{(1)}(x,y) = y^{-1} f_i(xy, y).$$

Observe that $f_i^{(1)}$ is a root of the polynomial $\tilde{P}$ defined by (3.10). We have again $f_i^{(1)}(0,y) \equiv 0$, so we may define $f_i^{(2)}$ a strict transform of $f_i^{(1)}$, and so on.

So actually we have proved:

Lemma 3.9. *For each f_i there exists an integer N such that our strict transform $f_i^{(N)}$ is analytic in a neighbourhood of* $0 \in \mathbb{R}^2$.

Proof of Proposition 3.8. Now we shall prove that $y \mapsto \frac{\partial f_i}{\partial x}(0,y)$ is analytic at $0 \in \mathbb{R}$. Let us expand f_i as a power series in x,

$$f_i(x,y) = \sum_{n=0}^{\infty} c_n^i(y) x^n.$$

We have to prove that $\frac{\partial f_i}{\partial x}(0,y) = c_1^i(y)$ is analytic at $0 \in \mathbb{R}$.

By the change of variable $z \mapsto z - c_0^i(y)$, we may assume that $f_i(0,y) \equiv c_0^i(y) \equiv 0$. Let us compute a proper transform of f_i:

$$\frac{f_i(xy, y)}{y} = \sum_{n=1}^{\infty} c_n^i(y) y^{n-1} x^n.$$

So the coefficient $c_1^i(y)$ remains unchanged ! We know, by Lemma 3.9 that our strict transform $f_i^{(N)}$ is analytic in a neighbourhood of $0 \in \mathbb{R}^2$.

Thus $c_1^i(y)$ is a partial derivative of an analytic function $f_i^{(N)}$, hence it is analytic itself.

4. Roots of hyperbolic polynomials are lipschitz

We answer positively a question asked by S.Łojasiewicz. First we introduce some notations. Consider a polynomial

$$P(x,z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i},$$

with $a_i : \Omega \to \mathbb{R}$ real analytic functions in an open set $\Omega \subset \mathbb{R}^n$. Assume that for each $x \in \Omega$ all the roots of the polynomial $z \mapsto P(x,z)$ are real; we denote them by $\lambda_1(x) \leq \cdots \leq \lambda_d(x)$. So we have a mapping $\Lambda : \Omega \to \mathbb{R}^d$ defined by

$$\Lambda(x) = (\lambda_1(x), \ldots, \lambda_d(x)). \tag{4.1}$$

By a classical result we know that Λ is continuous (see eg. [5], [23]). But of course Λ is not analytic; take for instance $z^2 - x^2$, then $\lambda_1(x) = -|x|$ and $\lambda_2(x) = |x|$. If $n = 1$, then by Rellich's Theorem 3.4 we can write the components of Λ as a MinMax of a family of d analytic functions. But this no longer possible if $n \geq 2$, consider $z^2 - (x_1^2 + x_2^2)$. However this example suggests that Λ is more then merely continuous and S.Łojasiewicz asked whether Λ is locally lipschitz. Indeed this is the case. We will prove the following.

Theorem 4.1. *The mapping $\Lambda : \Omega \to \mathbb{R}^d$ is locally lipschitz.*

This result is quite delicate as shown by several examples of arc-analytic functions which are not lipschitz (see Section 2). The proof of the theorem will be given in the next section. We now relate our theorem to some known facts in the literature.

4.1. **Lidskii's theorem; hyperbolic polynomials versus symmetric matrices.** Let $\mathcal{S}_d$ denote the space of $d \times d$ symmetric matrices with real coefficients. Recall that $\dim \mathcal{S}_d = \frac{d(d+1)}{2}$. We have a canonical analytic map

$$\theta : \mathcal{S}_d \to \mathcal{P}_d$$

which associate to a matrix $A \in \mathcal{S}_d$ its characteristic polynomial $\theta(A)$. Here $\mathcal{P}_d$ stands for the space of monic polynomials of degree d. We identify a vector in $\mathbb{R}^d$ with a monic polynomial of degree d as in Section 3. Let us denote by $\mathcal{H}_d = \theta(\mathcal{S}_d)$ the space of hyperbolic polynomials. $\mathcal{H}_d$ is semialgebraic and can be explicitly described by inequalities involving subresultants [5]. Actually, $\mathcal{H}_d$ is the closure of a connected component of the complement of the discriminant. Geometry of $\mathcal{H}_d$ was studied by Arnold [2], Givental [11], Kostov [15] and others. Its boundary is concave and piecewise lipschitz. We have another canonical map

$$\bar{\Lambda} : \mathcal{S}_d \to \mathbb{R}^d,$$

which associate to a matrix $A \in \mathcal{S}_d$ its eigenvalues in the increasing order, as in (4.1). There is a classical result, known as Lidskii's Theorem, which asserts the following.

Theorem 4.2. **(Lidskii 1950)** *Given two symmetric matrices $A, B \in \mathcal{S}_d$ then*

$$(\bar{\Lambda}(A) - \bar{\Lambda}(B)) \subset conv\{\tau(\bar{\Lambda}(A-B)) : \tau \in \mathcal{B}_d\},$$

where $\mathcal{B}_d$ stands for the group of permutations of the d coordinates and "conv" for the convex hull.

Lidskii's Theorem is not trivial at all, for proofs see [14],[8]. In particular it implies the following.

Corollary 4.3. *The mapping $\bar{\Lambda} : \mathcal{S}_d \to \mathbb{R}^d$ is globally lipshitz, (with an explicit constant).*

Note that our Theorem 4.1 implies that $\bar{\Lambda} : \mathcal{S}_d \to \mathbb{R}^d$ is locally lipschitz. Indeed we can write $\bar{\Lambda} = \Lambda \circ \theta$, in other words we can consider the analytic (in fact polynomial) family of characteristic polynomials parametrized by all symmetric matrices. However note that Lidskii's theorem does not imply our Theorem. Actually there are analytic families of hyperbolic polynomials which are not associated to an analytic family of symmetric matrices. More precisely if $P : \Omega \to \mathcal{P}_d$ is an analytic mapping then, in general, there is no analytic mapping $A : \Omega \to \mathcal{S}_d$ such that $P(x)$ is the characteristic polynomial of $A(x)$ for any $x \in \Omega$. Of course this is true if $\Omega \subset \mathbb{R}$; by Rellich's theorem, it is enough to take as $A(x)$ a diagonal matrix with the roots of $P(x)$ on diagonal.

F. Rellich observed in his book [28], Chapter I, Section 2, the following: let a_2 be an analytic function, then the polynomial

$$P(x, z) = z^2 - a_2(x)$$

is hyperbolic if $a_2(x) \geq 0$. Assume that $P(x, z)$ is a characteristic polynomial of an analytic family of matrices

$$\begin{pmatrix} a(x) & b(x) \\ b(x) & -a(x) \end{pmatrix}.$$

It follows that $a_2(x) = a(x)^2 + b(x)^2$. Rellich proved that any positive analytic function in 2 variables is a sum of 2 squares of analytic functions. But he also showed that in general a positive analytic function in 3 variables is not a sum of 2 squares of analytic functions. This is related to the Hilbert's 17th problem.

Remark 4.4. Our Theorem gives a locally lipschitz section $\lambda : \mathcal{H}_d \to \mathcal{S}_d$ of

$$\theta : \mathcal{S}_d \to \mathcal{H}_d.$$

5. Proof of theorem 4.1

We show that Theorem 4.1 follows from Proposition 3.8.

We will show that the components $\lambda_i(x)$ of $\Lambda : \Omega \to \mathbb{R}^d$ are locally lipschitz. That is; for any point $x_0 \in \Omega \subset \mathbb{R}^n$ there exists $r = r(x_0) > 0$ and $L = L(x_0) < \infty$ such that, if $|x - x_0| < r$ and $|y - x_0| < r$, then

$$|\lambda_i(x) - \lambda_i(y)| \leq L|x - y|, \; i = 1, \dots, d. \tag{5.1}$$

Recall that λ_i are C^1 in Ω, except a nowhere dense, analytic subset set $A \subset \Omega$. We are going to prove that each $\frac{\partial \lambda_i}{\partial x_k}$ is bounded in a neighbourhood of x_0, more precisely at points where λ_i is C^1, that is outside the analytic set A. Assume that this not the case for $\frac{\partial \lambda_i}{\partial x_1}$. Note that λ_i has semi-analytic graph, and then by the curve selection lemma (conform for instance [4], [22]) it follows that there exists an analytic arc $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that

$$\gamma(0) = x_0, \; |\frac{\partial \lambda_i}{\partial x_1}(\gamma(s))| \to \infty, \;\; as\, s \to 0.$$

Let $e_1 = (1, 0, \ldots, 0)$, and consider the mapping $g(s,t) = \gamma(s) + te_1$ and the associated hyperbolic polynomial $(s,t,z) \mapsto Q(s,t,z) = P(g(s,t),z)$. According to the Proposition 3.7, it splits in a horned neighbourhood of s-axis into $\prod_{i=1}^{d}[z - g_i(s,t)]$ with g_i analytic in that neighbourhood. So by Proposition 3.8

$$\frac{\partial \lambda_i}{\partial x_1}(\gamma(s)) = \frac{\partial g_i}{\partial t}(s,0)$$

is bounded for $s \to 0$. This is a contradiction, hence Theorem 4.1 follows.

6. Roots of hyperbolic polynomials as multivalued arc-analytic functions

In this section we prove that the roots of hyperbolic polynomials can be desingularized by sequences of blowing-ups with smooth centers. First we recall some known facts from algebra.

6.1. **Generalized discriminants.** Consider a generic polynomial

$$P_c(z) = z^d + c_1 z^{d-1} + \cdots + c_d$$

for $z \in \mathbb{C}$ and $c = (c_1, \ldots, c_d) \in \mathbb{C}^d$. We put

$$W_s = \{c \in \mathbb{C}^d : \ P_c(z) \text{ has at most } s \text{ distincts roots } \}.$$

Let $K = \{1, \ldots, d\}$ and put

$$\mathcal{D}_s(z_1, \ldots, z_d) = \sum_{J \subset K; \#J = d-s} \ \prod_{\mu,\nu \in J; \mu < \nu} (z_\mu - z_\nu)^2 \qquad , \quad s = 0, \ldots, d-1.$$

Since $\mathcal{D}_s(z_1, \ldots z_d)$ is a symmetric polynomial, we have $\mathcal{D}_s = D_s \circ \sigma$ with $\sigma = (\sigma_1, \ldots, \sigma_d)$, where $\sigma_1, \ldots, \sigma_d$ are the basic symmetric polynomials (by the well known theorem on symmetric functions). So D_s is a polynomial in $c = (c_1, \ldots, c_d)$. We shall call the sequence $D_s(c)$, $s = 0, \ldots, d-1$ the *generalized discrminants* of the polynomial P_c. By a similar theory of subresultants (see eg. [5]) we can find an explicit expression for $D_s(c)$ as a minor of the Sylvester matrix of P_c and P_c'. Note that $D_0(c)$ is the discriminant of P_c.

Lemma 6.1. *For $s = 0, \ldots, d-1$ we have*

$$W_s = \{c \in \mathbb{C}^d : D_0(c) = \cdots = D_{d-s-1}(c) = 0\}.$$

Indeed, if $c \in W_s$ and $z = (z_1, \ldots, z_d)$ is the complete sequence of roots of $P_c(z)$, then $\#\{z_1, \ldots, z_d\} \leq s$, hence $\mathcal{D}_0(z) = \cdots \mathcal{D}_{d-s-1}(z) = 0$; which implies

$$D_0(c) = \cdots = D_{d-s-1}(c) = 0.$$

Conversely, let $c \in \mathbb{C}^d$ be such that $D_0(c) = \cdots = D_{d-s-1}(c) = 0$ and let $z = (z_1, \ldots, z_d)$ the complete sequence of roots of $P_c(z)$. Assume that $c \notin W_s$, hence $s+1 \leq \#\{z_1, \ldots, z_d\} = t$. Let $z_1, \ldots z_t$ be the distinct roots of $P_c(z)$. So

$$\mathcal{D}_j(z_1, \ldots, z_d) = D_j(c) = 0 \qquad \text{for} \qquad j = 0, 1, \ldots d-s-1.$$

Since $d - t \leq d - s - 1$,

$$0 = \mathcal{D}_{d-t}(z_1, \ldots, z_d) = \prod_{\mu < \nu, \mu, \nu \in \{1, \ldots, t\}} (z_\mu - z_\nu)^2\,,$$

which is a contradiction. By the same argument we obtain the following.

Corollary 6.2. *Assume that P_c has exactly s distinct roots $z_1, \dots z_s$. Denote by*

$$\tilde{P}_c(z) = \prod_{i=1}^{s} (z - z_i)$$

a square-free polynomial which has the same roots as P_c, and by $D\tilde{P}_c$ the discriminant of $\tilde{P}_c$. Then

$$\nu_1 \cdots \nu_s D\tilde{P}_c = D_{d-s}(c),$$

where each ν_i is the multiplicity of z_i as a root of P_c.

In particular we can check whether $D\tilde{P}_c \neq 0$ without computing the coefficients of $\tilde{P}_c$.

6.2. **Splitting according to multiplicities of roots.** Consider polynomials of the form

$$P(x, z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i},$$

with a_i holomorphic functions in an open connected subset U of $\mathbb{C}^n$ (or more generally in a connected holomorphic manifold U). Recall that $\mathcal{M}(U)$, the ring of meromorphic functions on U is actually a field. So in the ring of polynomials $\mathcal{M}(U)[z]$ we have well defined *gcd* (greatest common divisor) of any finite family of polynomials in $\mathcal{M}(U)[z]$. In particular, if $P, Q \in \mathcal{M}(U)[z]$ are monic polynomials with holomorphic coefficients, then $R = gcd(P, Q)$ is again a monic polynomial with holomorphic coefficients. Indeed if we assume that R is monic (and $\deg R \geq 1$) then a priori the coefficients of R are only meromorphic, but the zeros of R are contained in zeros of P which are locally bounded (as multivalued functions of x). So the coefficients of R, being bounded and meromorphic, are actually holomorphic.

We shall say that P is *square-free* if its discriminant

$$DP(x) = D_0(a_1(x), \dots, a_d(x)) \not\equiv 0.$$

Recall that $DP : U \to \mathbb{C}$ is a holomorphic function. Of course each polynomial in $\mathcal{M}(U)[z]$ has a unique (up to a permutation) decomposition into irreducible factors.

We shall need the following splitting.

Proposition 6.3. *Let U be an open connected subset of $\mathbb{C}^n$ (or more generally a connected holomorphic manifold). Let*

$$P(x, z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i},$$

be a polynomial with a_i holomorphic in U.

Then there are unique (up to permutation) square-free monic polynomials $P_1, \dots, P_k$ with coefficients holomorphic in U and pairwise distinct integers $\nu_1, \dots, \nu_k \geq 1$, such that

$$P = P_1^{\nu_1} \cdots P_k^{\nu_k}. \tag{6.1}$$

Moreover $P_1, \dots, P_k$ are relatively prime; that is if $i \neq j$ then $gcd(P_i, P_j) = 1$.

Proof. Let $P' = \frac{\partial P}{\partial z}$. If P is not square-free then $DP = 0$ in $\mathcal{M}(U)$, so $R = gcd(P, P')$ is of degree at least 1. Hence $P = RQ$, where Q is a monic polynomial with holomorphic coefficients in U. But $\deg R < d$ and $\deg Q < d$, so it is easy to conclude applying induction on degree to R and Q. Alternatively we can decompose

$$P = Q_1^{m_1} \cdots Q_l^{m_l}, \tag{6.2}$$

where Q_j are irreducible. Now for a fixed integer $\nu_i \in \{m_1, \ldots, m_l\}$ we write P_i as the product of those irreducible factors of P which appear in (6.2) with exponent ν_i. $\square$

We will denote by $\tilde{P} = P_1 \cdots P_k$ the associate square-free polynomial. Of course we have also $\tilde{P} = Q_1 \cdots Q_l$. Clearly $P^{-1}(0) = \tilde{P}^{-1}(0)$. It follows from Corollary 6.2 that we can compute the discriminant $D\tilde{P}$ without performing the splitting (6.1). Precisely,

Corollary 6.4. *Assume that $P(x, z)$ is as in Proposition 6.3. Let $s = \sum_{i=1}^{k} \deg P_i$. Then*

$$\nu_1 \cdots \nu_s D\tilde{P}(x) = D_{d-s}(a_1(x), \ldots, a_d(x)) \not\equiv 0.$$

Moreover for each $x \in U$ the polynomial $z \mapsto P(x, z)$ has at most s distinct roots and if $D\tilde{P}(x) \neq 0$, then it has exactly s distinct roots.

6.3. **Quasi-ordinary singularities.** Let U be an open subset of $\mathbb{C}^n$ (or more generally a holomorphic manifold). Let

$$P(x, z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i},$$

be a polynomial with a_i holomorphic in U. We say that P is *quasi-ordinary* if the discriminant $D\tilde{P}$ of the square-free reduction $\tilde{P}$ of P is a normal crossing. In other words for each $a \in U$ there exists a local chart around a such that $D\tilde{P}(x) = u(x) x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, with $u(a) \neq 0$.

The concept of quasi-ordinary singularities goes back (at least) to Jung's (1908) desingularization of embedded algebraic surfaces. In fact they appear as "terminal" singularities which can be resolved by the normalization. We shall need a crucial property which generalizes the Newton-Puiseux parametrization. The result below is sometimes called Abhyankar-Jung theorem.

Theorem 6.5. (Jung 1908) *Let $U = \{|x_1| < r_1\} \times \cdots \times \{|x_n| < r_n\}$ be an open polydisc in $\mathbb{C}^n$ and let*

$$P(x, z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i},$$

be a polynomial with a_i holomorphic in U. Assume that the discriminant $D\tilde{P}$ of the square-free reduction $\tilde{P}$ of P is of the form $D\tilde{P}(x) = u(x) x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $u(x)$ is a holomorphic non-vanishing function in U. Then there exist integers $q_1, \ldots, q_n \geq 1$ and holomorphic functions $f_1, \ldots, f_d$ defined in the polydisc $U' = \{|z_1| < r_1^{1/q_1}\} \times \cdots \times \{|z_n| < r_n^{1/q_n}\}$ such that

$$P(x_1^{q_1}, \ldots, x_1^{q_n}, z) = \prod_{i=1}^{d} (z - f_i(x_1, \ldots, x_n)) \tag{6.3}$$

for any $(x_1, \ldots, x_n) \in U'$, $z \in \mathbb{C}$.

See for instance [29], [21] or [3].

6.4. **Splitting of quasi-ordinary hyperbolic polynomials.** We formulate now an important consequence of Jung's theorem for hyperbolic polynomials.

Proposition 6.6. *Let $\Omega = (-r, r)^n$ be an open cube in $\mathbb{R}^n$ and let*

$$P(x, z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i},$$

be a hyperbolic polynomial with a_i analytic in Ω. Assume that the discriminant $D\tilde{P}$ of the square-free reduction $\tilde{P}$ of P is of the form $D\tilde{P}(x) = u(x) x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $u(x)$ is analytic and non vanishing in Ω. Then there exist analytic functions $f_1, \ldots, f_d : \Omega \to \mathbb{R}$ such that

$$P(x, z) = \prod_{i=1}^{d} (z - f_i(x)) \tag{6.4}$$

for any $x \in \Omega$, $z \in \mathbb{C}$.

Proof. Note that it is enough to prove the result in a neighbourhood of any point of Ω and then use the uniqueness of analytic extension to obtain functions $f_1, \ldots, f_d$ defined in Ω. So we may assume that the coefficients a_i are actually holomorphic in the polydisc $U = \{|x_1| < r\} \times \cdots \times \{|x_n| < r\}$ in $\mathbb{C}^n$. We may also assume that the unit $u(x)$ does not vanish in U. We apply Jung's Theorem 6.5. Let us take the smallest integers $q_1, \ldots, q_n \geq 1$ so that there are analytic functions $f_1, \ldots, f_n$ such that formula (6.3) holds. We claim that actually $q_1 = \cdots = q_n = 1$.

Assume that one $q_i > 1$, for instance that $q_1 > 1$. We expand f_i as a power series in x_1 with coefficients holomorphic in $x' = (x_2, \ldots, x_n)$. Let us fix an i and write f instead of f_i. So we have

$$f(x_1, x') = \sum_{\nu=0}^{\infty} c_\nu(x') x_1^\nu \tag{6.5}$$

Since $q_1 > 1$ is minimal, then there exists $\nu_0 \in \mathbb{N} \setminus q_1\mathbb{N}$ such that $c_{\nu_0} \not\equiv 0$. So there exists $a' \in (-r, r)^{n-1} \subset \mathbb{R}^{n-1}$, such that $c_{\nu_0}(a') \neq 0$. Hence the Puiseux expansion of the function $g(x_1) = f(x_1^{\frac{1}{q_1}}, a')$, $x_1 > 0$ has at least one monomial with non-integer exponent. So $g(x_1)$ cannot be extended to an analytic function in a neighbourhood of $0 \in \mathbb{R}$. But on the other hand, by Rellich's Theorem 3.4, the roots of the polynomial $P(x_1, a', z)$ are analytic functions on $(-r, r)$. Clearly g must be a restriction of one of these functions. This is a contradiction.

□

As a first consequence of Proposition 6.6 observe that the roots of an analytic family of hyperbolic polynomials can be chosen analytically outside a subset of codimension at least two.

Theorem 6.7. *Consider a polynomial*

$$P(x,z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i},$$

where $a_i : \Omega \to \mathbb{R}$ are real analytic functions in an open set $\Omega \subset \mathbb{R}^n$. Assume that for each $x \in \Omega$ all roots of the polynomial $z \mapsto P(x,z)$ are real.

Then, there exists $\Sigma \subset \Omega$ a semianalytic closed set of codimension at least 2 such that if $a \in \Omega \setminus \Sigma$ then there is a neighbourhood U of a and analytic functions $f_i : U \to \mathbb{R}$, $i = 1, \dots, d$ such that

$$P(x,z) = \prod_{i=1}^{d} (z - f_i(x)),$$

for any $x \in U$, $z \in \mathbb{R}$.

Proof. Without loss of generality we may assume that Ω is connected. So the discriminant $D\tilde{P}$ of the square-free reduction $\tilde{P}$ of P is a well defined non vanishing analytic function on Ω. We are going to prove that $D\tilde{P}$ is a normal crossing outside a closed semianalytic set of codimension at least 2.

Let Z be the set of zeros of $D\tilde{P}$. Clearly Z is an analytic subset of Ω. Let $Reg_{n-1}Z$ be the set of points $x \in \Omega$ such that for some neighbourhood U of x the set $Z \cap U$ is an analytic submanifold of dimension $n-1$. Of course $Reg_{n-1}Z$ is open in Z and by Łojasiewicz's theorem [22],

$$\Sigma' = Z \setminus Reg_{n-1}Z,$$

is a semianalytic set, $\dim \Sigma' < \dim Z \leq n-1$. Hence Σ' is closed in Ω of codimension at least 2. Let Δ be a connected component of $Reg_{n-1}Z$. Let α be the smallest integer such that

$$h = \frac{\partial^\alpha D\tilde{P}}{x_1^{r_1} \dots x_n^{r_n}}$$

does not vanish identically on Δ for some multi index $(r_1, \dots, r_n)$, $\alpha = r_1 + \dots + r_n$. Hence $\Sigma''(\Delta) = h^{-1}(0) \cap \Delta$ is a semianalytic set of dimension less than $(n-1)$. Note that if $a \in \Delta \setminus \Sigma''(\Delta)$ then in some chart around a we can write

$$D\tilde{P}(x_1, \dots, x_n) = u(x) x_1^\alpha,$$

with some unit $u(x)$. Let

$$\Sigma'' = \bigcup \Sigma''(\Delta),$$

where the union is taken over all connected components of $Reg_{n-1}Z$. Finally we put

$$\Sigma = \Sigma' \cup \Sigma''.$$

Clearly Σ is semianalytic and closed in Ω of codimension at least 2. Let $a \in \Omega \setminus \Sigma$, then $D\tilde{P}$ is a normal crossing in a neighborhood of a so we conclude using Proposition 6.6. $\square$

Remark 6.8. In particular if $\dim M = 2$ then Σ has only isolated points. In other words any 2-parameter analytic family of hyperbolic polynomials splits locally, outside a discrete set, into linear factors. More generally, observe that if Ω is connected and Σ is semianalytic closed in Ω of codimension at least 2, then $\Omega \setminus \Sigma$ is also connected. However we cannot

claim that we can split $P(x,z)$ into a product of linear factors in $\Omega \setminus \Sigma$. Here we may have a nontrivial monodromy.

Example 6.9. The discriminant of the hyperbolic polynomial

$$P(x,y,z) = z^3 - 3(x^2+y^2)z - 2x^3,$$

vanishes on the x-axis. Here Σ is just the origin. Note that $P(x,0,z) = (z+x)^2(z-2x)$. So for $x>0$ the double root is smaller than the simple root, while for $x<0$ their order is inversed. Moving around the circle $\{x^2+y^2=1\}$ in $\mathbb{R}^2$ gives a nontrivial monodromy.

6.5. **Multivalued arc-analytic functions.** For the purpose of studying hyperbolic polynomials we use the following notion. Let M^m, N^n be two real analytic manifolds. Let F be a subanalytic subset of $M \times N$. For $x \in M$ we denote

$$F(x) = \{y \in N : \ (x,y) \in F\}$$

and we call $F(x)$ the *set of values of F at x*. If $F(x)$ is non empty for every $x \in M$ we say that F is *a multivalued mapping on M with values in N*. If M is connected we say that F is *k-valued* if $F(x)$ has at most k points for any $x \in M$ and exactly k points for some $x_0 \in M$. Single valued F is a function in the usual sense.

We will say that F is *continuous* if F is closed in $M \times N$. We call F *proper* if the projection on M restricted to F is a proper map. We say that $F \subset M \times N$ is a *k-valued arc-analytic mapping* if for any analytic arc $\gamma : (-\varepsilon, \varepsilon) \to U \subset M$ there are k analytic functions $f_i : (-\varepsilon,\varepsilon) \to N$, $i = 1, \ldots, k$ such that

$$F(\gamma(t)) = \{f_1(t), \ldots, f_k(t)\}.$$

Note that, in general, the set $\{f_i(t)\}$ is **not ordered**. If F is single valued then it is an arc-analytic mapping in the usual sense.

Theorem 6.10. *Every proper k-valued arc-analytic and subanalytic mapping is locally blow-analytic via a locally finite collection of analytic modifications $\sigma_\alpha : W_\alpha \to M$, that is for any σ_α we have*

$$\tilde\sigma_\alpha^{-1}(F) = \bigcup \tilde F_i$$

and each $\tilde F_i$ is a graph of an analytic function in $\sigma_\alpha^{-1}(M)$. Here $\tilde\sigma_\alpha : W_\alpha \times N \to M \times N$, $\tilde\sigma_\alpha(w,y) = (\sigma_\alpha(w), y)$. If F is semialgebraic then, instead of the family σ_α we can take one σ which is a finite composition of global blowing-ups with smooth centers.

The proof of this result will be published separately.

In the case of hyperbolic polynomials with analytic coefficients the Theorem 6.10 can be restated as follows.

Theorem 6.11. *Consider a polynomial*

$$P(x,z) = z^d + \sum_{i=1}^{d} a_i(x) z^{d-i},$$

where $a_i : \Omega \to \mathbb{R}$ are real analytic functions in an open set $\Omega \subset \mathbb{R}^n$. Assume that for each $x \in \Omega$ all the roots of the polynomial $z \mapsto P(x,z)$ are real. Then, there exists $\sigma : W \to \Omega$ a locally finite composition of blowing-ups with smooth (global) centers, such that for any $w_0 \in W$ there are a neighbourhood U and analytic functions $F_i : U \to \mathbb{R}$, $i = 1, \dots, d$ such that

$$P_\sigma(w,z) = z^d + \sum_{i=1}^{d} a_i(\sigma(w))z^{d-i} = \prod_{i=1}^{d}(z - F_i(w)),$$

for any $w \in U$, $z \in \mathbb{R}$.

Remark 6.12. Note that the Theorem 6.11 applies also to real analytic families of monic polynomials such that for each $x \in \Omega$ all the roots of the polynomial $z \mapsto P(x,z)$ are purely imaginary. Indeed, if $P(x,z)$ is such a polynomial then $P(x,iz)$ is hyperbolic with real analytic coefficients.

Proof. Without loss of generality we may assume that Ω is connected. So the discriminant $D\tilde{P}$ of the square-free reduction $\tilde{P}$ of P is a well defined non vanishing analytic function on Ω. By Hironaka's Desingularization Theorem [12], there exists $\sigma : W \to \Omega$ a locally finite composition of blowing-ups with smooth (global) centers, such that $D\tilde{P} \circ \sigma$ is a normal crossing. But $D\tilde{P} \circ \sigma$ is the discriminant of the square-free reduction of $P_\sigma(w,z) = z^d + \sum_{i=1}^{d} a_i(\sigma(w))z^{d-i}$. So the theorem follows immediately from Proposition 6.6. $\square$

Example 6.13. Let $P(z,x_1,x_2) = z^2 - (x_1^2 + x_2^2)$, so $DP = 4(x_1^2 + x_2^2)$ is the discriminant of P. Clearly the blowing-up of the origin makes it normal crossing. Namely, we write $x_1 = w_1$,$x_2 = w_1w_2$ for the blowing-up, so

$$P(w_1,w_2,z) = (z - F_1(w))(z - F_2(w)),$$

where $F_1 = w_1(1+w_2^2)^{1/2}$, $F_2 = -w_1(1+w_2^2)^{1/2}$ are real analytic functions (defined in one chart). Note that these functions are not holomorphic if we consider w_1, w_2 as complex numbers. This simple example shows the purely real character of Theorem 6.11.

In the sequel we shall need that the analytic functions $F_i : U \to \mathbb{R}$ in Theorem 6.11 are also normal crossings. This can be achieved by making the coefficient a_d normal crossing, indeed we have $a_d \circ \sigma = F_1 \cdots F_d$. Of course each factor of a normal crossing is again a normal crossing. Assume that we arranged F_i's in such a way that $F_1, \dots, F_s$ are all the distinct roots of $P_\sigma(w,z)$. Recall that,

$$D\tilde{P} \circ \sigma(w) = \prod_{i<j\le s}(F_i(w) - F_j(w))^2.$$

But $D\tilde{P} \circ \sigma(w)$ is a normal crossing so it follows that for any $i,j \le s$, $i \ne j$ the function $(F_i(w) - F_j(w))$ is a normal crossing as well. Recall now a very important observation in Bierstone and Milman [4]:

Lemma 6.14. *Let U be an open connected subset of $\mathbb{R}^n$. Let $F_i \not\equiv 0$, $i = 1, \dots, d$ be analytic functions in U. Assume that all F_i and all their differences $F_i - F_j$ are normal crossings (or identically 0). Then for each $w \in U$ the exists a neighbourhood U_w such that for any $i,j \le d$ at least one of the functions $\frac{F_i}{F_j}$ or $\frac{F_j}{F_i}$ extends to an analytic function in U_w. In particular*

there exists $i_w \leq d$ such that $\frac{F_j}{F_{i_w}}$ extends to an analytic function in U_w, for any j. We will say for short that $F_1, \dots, F_d$ are **well ordered on** U_w.

As consequence we obtain:

Remark 6.15. In Theorem 6.11 we can choose U and the analytic functions $F_i : U \to \mathbb{R}$ in such a way that they are well ordered on U.

7. Diagonalization of analytic families of symmetric matrices

The goal of this section is to generalize a result of Rellich [28] which states that a 1-parameter analytic family of symmetric matrices admits a uniform diagonalization.

We will denote by $\mathcal{S}_d$ the space of symmetric $d \times d$ matrices with real entries. We consider first an analytic family of symmetric matrices $A : \Omega \to \mathcal{S}_d$, where Ω is an open connected subset of $\mathbb{R}^n$. Assume that the eigenvalues of $A(x)$ can be chosen analytically in Ω. Precisely we assume that there are analytic functions $F_i : \Omega \to \mathbb{R}$, $i = 1, \dots, d$ such that $\{F_1(x), \dots, F_d(x)\}$ is the set of the eigenvalues of $A(x)$, $x \in \Omega$. For a generic point $x \in \Omega$ (i.e. outside a nowhere dense analytic subset) each $F_i(x)$ is of the same constant multiplicity ν_i. If an eigenvalue $F_i(x)$ is actually of a constant multiplicity on Ω then, for any $x \in \Omega$

$$V_i(x) = Ker(A(x) - F_i(x)\mathbb{I}_d)$$

is an analytic family of ν_i-dimensional eigenspaces of $A(x)$. In particular we can choose locally, in an analytic way, an orthonormal basis of $V_i(x)$. However in general at some points the dimension of $V_i(x)$ may be strictly greater than ν_i. If $\Omega = I$ is an interval in $\mathbb{R}$ and $x_0 \in I$ is such that $\dim V_i(x_0) > \nu_i$, then $\lim_{x \to x_0} V_i(x)$ exists, in the corresponding Grassmanian. Moreover the mapping $x \mapsto V_i(x)$ obtained by the continuous extension is actually analytic (as we will show later on). However this is no longer true if we consider an analytic family depending on $n \geq 2$ parameters. Indeed we have,

Example 7.1. Consider a family of symmetric matrices of the form

$$A(x_1, x_2) = \begin{pmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{pmatrix}, \ (x_1, x_2) \in \mathbb{R}^2.$$

Note that $\phi = 0$, $\psi = x_1^2 + x_2^2$ are eigenvalues of $A(x_1, x_2)$ and $\Phi = (1, \frac{x_2}{x_1})$,$\Psi = (1, -\frac{x_1}{x_2})$ are the corresponding eigenvectors. Clearly there is no limit of Φ and Ψ as $(x_1, x_2) \to (0, 0)$. So this family cannot be simultaneously diagonalized in an analytic (even continuous) way. However if we blow up the origin in $\mathbb{R}^2$, that is we put $x_1 = w_1$,$x_2 = w_1 w_2$, then the corresponding family

$$A(w_1, w_2) = w_1^2 \begin{pmatrix} 1 & w_2 \\ w_2 & w_2^2 \end{pmatrix}, \ (w_1, w_2) \in \mathbb{R}^2,$$

admits a simultaneous analytic diagonalization.

The next theorem explains that this happens for a general analytic family of symmetric matrices.

To fix the terminology we recall that if $\Phi : E \to E$ is a linear mapping and λ is an eigenvalue of Φ, then $E_\lambda = \{x \in E;\ \Phi(x) = \lambda x\}$ is called **the eigenspace** of Φ (associated to λ). Any nontrivial linear subspace of E_λ is called **an eigenspace** of Φ (associated to λ).

Theorem 7.2. *Consider an analytic family $A : \Omega \to \mathcal{S}_d$ of symmetric matrices, where Ω is an open connected subset of $\mathbb{R}^m$ and $\mathcal{S}_d$ stands for the space of symmetric $d \times d$ matrices with real entries. Then, there exists $\sigma : W \to \Omega$ a locally finite composition of blowing-ups with smooth (global) centers, such that for any $w_0 \in W$ there is a neighbourhood U such that the corresponding family $A \circ \sigma|_U : U \to \mathcal{S}_d$ admits a simultaneous analytic diagonalization.*

More precisely, let $P_\sigma(w, z)$ be the characteristic polynomial of $A \circ \sigma(w)$; recall that for a generic $w \in W$ the polynomial $P_\sigma(w, z)$ has s distinct real roots with the constant number of roots of fixed multiplicity. Then, for each $w \in W$, there exits an orthogonal decomposition

$$\mathbb{R}^d = V_1(w) \oplus \cdots \oplus V_s(w), \tag{7.1}$$

such that:

(i) *each $V_i(w)$ is an eigenspace of $A \circ \sigma(w)$, $\dim V_i(w) = m_i \geq 1$;*

(ii) *if the eigenvalue $\lambda_i(w)$ associated to $V_i(w)$ is a root of $P_\sigma(w, z)$ of multiplicity m_i then,*

$$Ker(A \circ \sigma(w) - \lambda_i(w)\mathbb{I}_d) = V_i(w);$$

(iii) *if the eigenvalue $\lambda_i(w)$ associated to $V_i(w)$ is a root of $P_\sigma(w, z)$ of multiplicity $> m_i$ then,*

$$Ker(A \circ \sigma(w) - \lambda_i(w)\mathbb{I}_d) = V_i(w) \oplus V_{i_1}(w) \oplus \cdots \oplus V_{i_k}(w)$$

for some $i_1, \dots, i_k \in \{1, \dots, s\} \setminus \{i\}$;

(iv) *for any $w_0 \in W$ there is a neighbourhood U and analytic functions $e_i : U \to (\mathbb{R}^d)^{m_i}$, $i = 1, \dots, s$ such that $e_i(w)$ is an orthonormal basis of $V_i(w)$.*

Remark 7.3. Theorem 7.2 holds also for real analytic families of Hermitian matrices.

Remark 7.4. We can describe a global structure of bundles given by (7.1). Recall that W is connected so the polynomial $P_\sigma(w, z)$ admits a unique decomposition into irreducible factors

$$P_\sigma = Q_1^{m_1} \cdots Q_l^{m_l}. \tag{7.2}$$

Let us fix one $Q_j(w, z)$ and write $m = m_j$. By Theorem 6.11 we may assume that in an open set $U \subset W$ we can choose roots of $Q_j(w, z)$ as analytic functions $\lambda_1, \dots \lambda_{d_j} : U \to \mathbb{R}$. Recall that for a generic $w \in U$ all $\lambda_i(w)$ are simple roots of $Q_j(w, z)$. Let us denote by $\mathbf{G}_d^m$ the Grassmanian of m-dimensional subspaces of $\mathbb{R}^d$. Then according to Theorem 7.2 we have orthogonal subspaces $V_1(w), \dots, V_{d_j}(w) \in \mathbf{G}_d^m$, $d_j = \deg Q_j$, which are proper subspaces of $A \circ \sigma(w)$ associated to $\lambda_1(w), \dots \lambda_{d_j}(w)$. Let us collect them together and write

$$\Xi_j = \bigcup_{w \in W} \{V_1(w), \dots, V_{d_j}(w)\} \times w \subset \mathbf{G}_d^m \times W.$$

Note that Ξ_j is an analytic submanifold of $\mathbf{G}_d^m \times W$, moreover the natural projection $\pi : \Xi_j \to W$ is an analytic d_j-sheeted covering. Indeed Ξ_j may be seen as a multivalued analytic function; each $U \ni w \mapsto V_i(w)$ is analytic, and the values are distinct since the subspaces are orthogonal. Finally we observe that,

Proposition 7.5. *Ξ_j is connected.*

Proof. Indeed, if Ξ is a connected component of Ξ_j, then $\pi : \Xi \to W$ is again an analytic p-sheeted covering. For each w_0 there is a neighbourhood U and p distinct analytic sections of $\pi : \Xi \to W$. To each section (of eigenspaces) we can associate the corresponding eigenvalues $\lambda_1, \dots \lambda_p : U \to \mathbb{R}$, which are analytic functions. We put

$$Q(w,z) = \prod_{i=1}^{p} (z - \lambda_i(w)),\ w \in U.$$

By connectedness we can extend analytically Q on W. Clearly Q divides Q_j but Q_j is irreducible, so $Q = Q_j$, hence $\Xi = \Xi_j$.

□

We begin now the proof of Theorem 7.2. Since the characteristic polynomial of a symmetric matrix is hyperbolic, by Theorem 6.11, there exists $\sigma : W \to \Omega$, a locally finite composition of blowing-ups with smooth (global) centers, such that the eigenvalues of the corresponding family $A \circ \sigma$ are locally normal crossings. We are going to construct $\sigma' : W' \to W$, a locally finite composition of blowing-ups with smooth (global) centers, such that the corresponding family $A \circ (\sigma \circ \sigma')$ admits a simultaneous analytic diagonalization.

Recall that we have the splitting of $P_\sigma = Q_1^{m_1} \cdots Q_l^{m_l}$ into irreducible factors, where P_σ is the characteristic polynomial of the family $A \circ \sigma$. Let us fix one $Q_j(w,z)$ and write $m = m_j$.

We shall first explain the simpler case where $m = 1$. Hence, for a generic $w \in U$, all eigenspaces associated to the roots of $Q_j(w,z)$ are of dimension 1. Let $\lambda : U \to \mathbb{R}$ be an analytic choice of roots of $Q_j(w,z)$, where U is an open neighbourhood of some fixed point $w_0 \in W$. The eigenspace of $A \circ \sigma(w)$ associated to $\lambda(w)$ is a set of solutions of a $d \times d$ system

$$(A \circ \sigma(w) - \lambda(w)\mathbb{I}_d)X = 0 \tag{7.3}$$

Recall that for a generic $w \in U$ this system is of rank $d-1$. So we can delete one equation from (7.3) and we obtain an equivalent system

$$B(w)X = 0, \tag{7.4}$$

where $B(w)$ is a matrix with $d-1$ rows and d columns. Let $M_k(w)$ denote the determinant of the $(d-1)\times(d-1)$ matrix obtained from $B(w)$ by deleting the k-th column. By Cramer's rule we obtain that

$$\bar{v}(w) = (-M_1(w), \dots, (-1)^k M_k(w), \dots, (-1)^d M_d(w))$$

is a solution of (7.3). But of course we have to check that $\bar{v}(w) \neq 0$, which is true for a generic $w \in U$, but in general not for all $w \in U$. In particular we might possibly have $\bar{v}(w_0) = 0$. So we want to divide all the coefficients of $\bar{v}(w)$ by one of them and get again analytic coefficients.

We may assume (we explain it below) that all minors M_k, $k = 1, \dots, d$ are normal crossings and moreover that they are well ordered at w_0 (cf. Lemma 6.14). Permuting, if necessary, the coordinates in $\mathbb{R}^d$ we may assume that $M_1(w)$ is the smallest among all M_k, $k = 1, \dots, d$. In other words

$$m_k(w) = (-1)^{k-1}\frac{M_k(w)}{M_1(w)}, \quad k = 2, \dots, d$$

extend to analytic functions in a neighbourhood of w_0. Thus

$$v(w) = (1, m_2(w), \ldots, m_d(w))$$

is actually an eigenvector of $A \circ \sigma(w)$ associated to $\lambda(w)$. Clearly $v(w)$ is analytic in a neighbourhood of w_0. Finally we normalize $v(w)$ in order to get an orthonormal basis of $V(w) = \mathbb{R}v(w)$, the subspace generated by $v(w)$. Note that, for a generic $w \in U$, we have $V(w) = Ker(A \circ \sigma(w) - \lambda(w)\mathbb{I}_d)$.

We consider now the general case where the factor Q_j appears with exponent $m = m_j \geq 1$. So now, for a generic $w \in U$, $\lambda(w)$ is a root of $Q_j(w, z)$ of multiplicity m. The eigenspace of $A \circ \sigma(w)$ associated to $\lambda(w)$ is a set of solutions of the $d \times d$ system

$$(A \circ \sigma(w) - \lambda(w)\mathbb{I}_d)X = 0 \tag{7.5}$$

and for a generic $w \in U$ this system is of rank $d-m$. We shall construct linearly independent $v_1(w), \ldots, v_m(w) \in \mathbb{R}^d$ which are analytic in a neighbourhood of w_0 and such that, for a generic w we have

$$V(w) = Ker(A \circ \sigma(w) - \lambda(w)\mathbb{I}_d),$$

where $V(w) = span(v_1(w), \ldots, v_m(w))$ stands for the subspace generated by $v_1(w), \ldots, v_m(w)$.

So we can delete m equations from (7.5) and we obtain an equivalent (generically) system

$$B(w)X = 0, \tag{7.6}$$

where $B(w)$ is a matrix with $d - m$ rows and d columns. As in the case $m = 1$ we can consider all $(d-m) \times (d-m)$ minors of $B(w)$ and we may assume that they are well ordered at w_0. Let $M(w)$ be the smallest (at w_0) among all these minors. Permuting, if necessary, the coordinates in $\mathbb{R}^d$ we may suppose that $M(w)$ is the determinant of the matrix $C(w)$ formed by the first $d - m$ columns. We construct a vector

$$v_1(w) = (a_1(w), 1, 0, \ldots, 0), \tag{7.7}$$

where $a_1(w) \in \mathbb{R}^{d-m}$ is the solution of a system

$$C(w)X' = b_{d-m+1}(w), \tag{7.8}$$

here $b_{d-m+1}(w)$ denotes the $(d - m + 1)$-column of $B(w)$. Observe that the coordinates of $a_1(w)$ are quotients of some minors of $B(w)$ by $M(w)$, so they extend to analytic functions in a neighbourhood of w_0. We construct $v_2(w), \ldots, v_m(w)$ analogously by shifting 1 to the right in (7.7) and considering next columns of $B(w)$ in (7.8). Finally to obtain an orthonormal basis of $V(w)$ we apply the Gram-Schmidt orthonormalization to the family $v_1(w), \ldots, v_m(w)$.

We are left with proving that after a suitable composition of blowing-ups with smooth global centers, the minors of $B(w)$ are normal crossings which are well ordered at any point of W. We begin with a lemma which is actually a description of the normaliztion of a zero set of an irreducible hyperbolic polynomial with discriminant which is a normal crossing.

Lemma 7.6. *Consider a hyperbolic polynomial $Q(x, z) = z^d + \sum_{i=1}^{d} a_i(x)z^{d-i}$, where $a_i : \Omega \to \mathbb{R}$ are real analytic functions in a connected analytic manifold Ω. Assume that Q is irreducible and moreover that the discriminant $DQ : \Omega \to \mathbb{R}$ is a normal crossing. Then*

there exist a connected analytic manifold Ξ and an analytic d-sheeted covering $p : \Xi \to \Omega$, an analytic function $z : \Xi \to \mathbb{R}$ such that

$$Q(p(\xi), z(\xi)) = 0, \quad \xi \in \Xi.$$

Proof. We define Ξ as a space of germs f_x, at points $x \in \Omega$, of analytic functions $f : U \to \mathbb{R}$ such that $Q(x, f(x)) = 0$, $x \in U$, where U are open subsets of Ω. We have also a canonical map $F : U \ni x \mapsto f_x \ni \Xi$, where f_x stands for the germ of f at the point x. These maps F define an analytic atlas on Ξ, thus we obtain a structure of an analytic manifold on Ξ. Note that we did not specify the topology on Ξ, but actually this is not necessary, see for instance [23]. We only have to check that the topology we obtain is Hausdorff, but this is the case since we consider only analytic functions.

Now the mapping $p : \Xi \to \Omega$ is defined, in the above chart, as the inverse of F, so clearly it is a local diffeomorphism. We put $z(\xi) = f(x)$, for $\xi = f_x$. It follows from Proposition 6.6 that $p : \Xi \to \Omega$ is indeed a d-sheeted covering. To prove that Ξ is connected we may use the same argument as in the proof of Proposition 7.5. $\square$

We come back to the proof of the fact that the minors of the matrices considered in (7.4) and (7.6) can be made (by a composition of suitable blowing-ups) well ordered normal crossings. Note that these minors can be seen as d-valued analytic functions on W. More precisely they extend to analytic functions $M_k : \Xi_j \to \mathbb{R}$ on the space Ξ_j associated to the polynomial Q_j, by Lemma 7.6. We denote by $p_j : \Xi \to W$ the corresponding covering. We will consider only those M_k, $k = 1, \ldots, K$ which are non identically zero. Recall that Ξ_j is connected hence these minors are non identically zero on any open subset of Ξ_j. So now we can associate to Q_j and $A \circ \sigma$ two analytic non identically zero functions $\Phi_j, \Psi_j : W \to \mathbb{R}$ defined as follows

$$\Phi_j(w) = \prod_{k=1}^{K} \prod_{\xi \in p_j^{-1}(w)} M_k(\xi) \tag{7.9}$$

and the function which is the product of differences of all factors in (7.9), that is

$$\Psi_j(w) = \prod (M_k(\xi) - M_{k'}(\xi')), \tag{7.10}$$

where the product is taken over all $k, k' \in \{1, \ldots, K\}$, $k \neq k'$ and $\xi, \xi' \in p_j^{-1}(w)$, $\xi \neq \xi'$. Finally we can take Φ and Ψ which are respectively the products of all Φ_j and Ψ_j associated to the prime factors of P_σ.

By Hironaka's desingularization theorem there exists $\sigma' : W' \to W$ which is a locally finite composition of blowing-ups with smooth global centers such that both $\Phi \circ \sigma'$ and $\Psi \circ \sigma'$ are normal crossings. Hence in particular each factor in (7.9) and (7.10) becomes a normal crossing.

Thus we achieved a proof of Theorem 7.2.

Remark 7.7. Note that in the proof of Theorem 7.2 we used not only the fact that the characteristic polynomials of symmetric matrices are hyperbolic but also the fact that the eigenspaces associated to different eigenvalues are orthogonal. Indeed we need to know that subspaces $V(w)$ and $V'(w)$ which are associated to generically different eigenvalues $\lambda(w)$ and $\lambda'(w)$ are orthogonal also for those w for which $\lambda(w) = \lambda'(w)$. This is the case by

continuity. However if we consider an analytic family of matrices which are diagonalizable over reals (but not symmetric), then it may happen that the subspaces $V(w)$ and $V'(w)$ which have trivial intersection for generic w may have nontrivial intersection for some w_0 such that $\lambda(w_0) = \lambda'(w_0)$. So Theorem 7.2 does not apply to such a family.

Example 7.8. The following one parameter family of diagonalizable matrices

$$\begin{pmatrix} 1-x^2 & x \\ 0 & 1+x^2 \end{pmatrix},\ x \in \mathbb{R}.$$

have eigenvectors $v_1(x) = (1, 0)$, $v_2(x) = (1, 2x)$, which form a basis of $\mathbb{R}^2$ except for $x = 0$. So we cannot choose a basis of eigenvectors in a continuous way.

8. Reduction of analytic families of antisymmetric matrices

The method of diagonalization of analytic families of symmetric matrices described in the previous section applies as well to analytic families of antisymmetric matrices. Indeed, the characteristic polynomial of an antisymmetric matrix has purely imaginary roots and it is easy to see (cf. Remark 6.12) that Theorem 6.11 applies also to real analytic families of monic polynomials with purely imaginary roots.

First we recall briefly some basic facts about antisymmetric matrices.

Lemma 8.1. *Let A be an antisymmetric $d \times d$ matrix with real coefficients, then:*

(i) *The eigenvalues of A are purely imaginary, moreover if μ is an eigenvalue of A then $-\mu$ is also an eigenvalue of A.*
(ii) *The eigenspaces associated to distinct eigenvalues are orthogonal.*
(iii) *There exists an orthogonal basis of $\mathbb{R}^d$ in which A has on the diagonal 0 or blocks of the form*
$$\begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix},$$
where $\lambda_k \in \mathbb{R}$ and $i\lambda_k$ is an eigenvalue of A. These are called canonical forms.
(iv) *An orthogonal basis of $\mathbb{R}^d$ for the canonical form can be constructed in the following way: for a fixed eigenvalue $\mu = i\lambda_k \neq 0$ we construct an orthogonal (orthonormal) basis $v_1, \ldots, v_r$ of the eigenspace (subspace of $\mathbb{C}^d$) associated to μ, then $\bar v_1, \ldots, \bar v_r$ is an orthogonal (orthonormal) basis of the eigenspace associated to $\bar\mu = -\mu$. We put*
$$e_k = \frac{1}{2}(v_k + \bar v_k), \quad f_k = \frac{i}{2}(v_k - \bar v_k). \tag{8.1}$$
Then $e_1, f_1, \ldots, e_r, f_r$ is an orthogonal (orthonormal) basis of a real subspace of $\mathbb{R}^d$ of dimension $2r$. In this basis A has the canonical form.

Theorem 8.2. *Consider an analytic family $A : \Omega \subset \mathbb{R}^m \to \mathcal{A}^d$, where $\mathcal{A}^d$ stands for the space of antisymmetric $d \times d$ matrices with real entries. Then, there exists $\sigma : W \to \Omega$ a locally finite composition of blowing-ups with smooth (global) centers, such that for any $w_0 \in W$ there is a neighbourhood U such that the corresponding family $A \circ \sigma|_U : U \to \mathcal{A}^d$ admits a simultaneous analytic reduction to the canonical form. That is, there exists an*

analytic choice of vectors $e : U \to (\mathbb{R}^d)^d$ *such that* $e(w)$ *is an orthonormal of basis of* $\mathbb{R}^d$ *and* $A(\sigma(w))$ *has on diagonal* 0 *or blocks of the form*

$$\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix},$$

for all $w \in U$.

Proof. The arguments are essentially the same as in the proof of Theorem 7.2. So we will sketch only the main lines of the proof. First we resolve the singularities of the discriminant of the characteristic polynomial $P(x, z)$, $x \in \Omega$ of our family. So we may assume that locally the roots of of $P(x, z)$ are analytic functions of x. Recall that if $\lambda(w)$ is such a root then $-\lambda(w)$ is also a root of $P(x, z)$. We construct (as in the solution of the system (7.5)) an orthonormal system of vectors $v_1(w), \ldots, v_r(w)$ which depends analytically on x, in such a way that, for a generic w, $V(w) = Ker(A \circ \sigma(w) - \lambda(w)\mathbb{I}_d)$, is generated by $v_1(w), \ldots, v_m(w)$. This requires of course blowing-ups of the space of parameters. Now by Lemma 8.1 vectors $\bar{v}_1(w), \ldots, \bar{v}_m(w)$ form an orthonormal basis of the eigenspace associated to $-\lambda(w)$. So applying formula (8.1) we obtain locally a canonical basis for $A(w)$ which depends analytically on w. $\square$

References

[1] D. Alekseevsky, A. Kriegl, M. Losik and P. WW. Michor, *Choosing roots of polynomials smoothly*, Israel Journal of Mathematics, **105** (1998), 203-233.

[2] V.I. Arnold, *Hyperbolic polynomials and Vandermonde mappings*, Funct. Anal. Appl. **20** (1986), 125-127.

[3] W. Barth, C. Peters, and A. Van de Ven,*Compact complexx surfaces*, Springer, 1984.

[4] E. Bierstone and P. D. Milman, *Semianalytic and Subanalytic sets*, Publ. I.H.E.S., **67** (1988), 5-42.

[5] R. Benedetti and J.-J. Risler,*Real algebraic and semialgebraic sets*, Hermann, 1990.

[6] E. Bierstone and P. D. Milman, *Arc-analytic functions*, Invent. math., **101** (1990), 411-424.

[7] E. Bierstone and P. D. Milman, A. Parusiński, *A function which is arc-analytic but not continuous*, Proc. Amer. Math. Soc., **113** (1991), 419-423.

[8] R. Bhatia, *Matrix Analysis*, Springer Verlag, 1997.

[9] T. Fukui, S. Koike and T.-C. Kuo, *Blow-analytic equisingularities, properties, problems and progress*, in "Real analytic and algebraic singularities", Pitman Research Notes in Mathematics Series, **381**, 1997, Longman, 8–29.

[10] T.Fukui, K.Kurdyka and L.Paunescu, *An inverse mapping theorem for arc-analytic homeomorphisms,* Banach Center Publications vol **65** (2004), 49–56.

[11] A.B. Givental, *Moments of random variables and the equivariant Morse lemma*, Russ. Math. Surveys 42,2 (1987), 275-276.

[12] H. Hironaka, *Resolution of Singularities of an algebraic variety over a field of characteristic zero, I-II* Ann. of Math., **97** (1964).

[13] H. Hironaka, *Introduction to real-analytic sets and real-analytic maps,* Quaderni dei Gruppi di Ricerca Matematica del Consiglio Nazionale delle Ricerche, Istituto Matematico "L. Tonelli" dell'Università di Pisa (1973).

[14] T.Kato, *Analytic perturbation theory*, Springer 1976.

[15] V.P. Kostov, *On the geometric properties of Vandermonde's mapping and on the problem of moments*, Proc. Roy. Soc. Edinburgh Sect. A **112**, 3-4 (1989), 203-211.

[16] T.-C. Kuo, *On classification of real singularities*, Invent. math., **82** (1985), 257–262.
[17] K. Kurdyka, *Ensembles semi-algébriques symétriques par arcs*, Math. Ann., **282** (1988), 445–462.
[18] K. Kurdyka, *A counterexemple to subanalyticty of an arc-analytic function*, Ann. Polon. Math. **55** (1991), 241-243.
[19] K. Kurdyka, *An arc-analytic function with nondiscrete singular set*, Ann. Polon. Math. **59,** 1 (1994), 251-254.
[20] K. Kurdyka and L. Paunescu, *Arc-analytic roots of analytic functions are lipschitz*, Proc. Am. Math. Soc., **132**, 6 (2004), 1693-1702.
[21] J. Lipman, *Introduction to resolution of singularities*, Proceedings of Symposia in Pure Mathematics **29** (1975), 187-230.
[22] S. Łojasiewicz, *Ensembles semi-analytiques,*, preprint, I.H.E.S. (1965).
[23] S.Łojasiewicz, *Introduction to complex analytic geometry*. Birkhäuser Veralg. (Basel), (1991).
[24] A. Parusiński, *Subanalytic functions* Trans. of the A.M.S. **344,** 2 (1994), 583–595.
[25] L. Paunescu, *An Implicit Function Theorem For Locally Blow-Analytic Functions*, Annales de l'Institut Fourier, Vol 51 (2001), no.4. pp 1089–1100.
[26] L. Paunescu, *An example of blow-analytic homeomorphism* in "Real analytic and algebraic singularities", Pitman Research Notes in Mathematics Series, **381**, 1997, Longman, 62–63.
[27] F. Rellich, *Störungstheorie der Spektralzerlegung, I*, Math. Ann. **113** (1937), 600-619.
[28] F. Rellich, *Perturbation theory of eigenvalue problem* Gordon and Breach, New York 1950.
[29] O. Zariski, *Algebraic Surfaces* Springer, New York 1971.

Laboratoire de Mathematiques (LAMA), Université de Savoie, UMR 5127 CNRS, 73-376 Le Bourget-du-Lac cedex FRANCE
E-mail address: Krzysztof.Kurdyka@univ-savoie.fr

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
E-mail address: laurent@maths.usyd.edu.au